"MCNAMARA'S 'DEONTIC DODECAGON FOR DWE' IN THE LIGHT OF OPPOSITIONAL GEOMETRY"

"O 'dodecágono deôntico para DWE' de McNamara sob a luz da Geometria Oposicional"

Alessio Moretti*

Abstract: Oppositional geometry, i.e. the study of the "oppositional figures", has long been approached in a rather random way. In deontic logic, beyond the classical "deontic square" (a particular application of the logical square, or square of opposition), this has given birth in 1972 to Kalinowski's "deontic hexagon" (a particular application of Jacoby's, Sesmat's and Blanché's "logical hexagon"), to Joerden's "deontic decagon" (1987), to McNamara's "deontic dodecagon" and "deontic octodecagon" (1996) and to Wessels' "deontic decagon" and "deontic hexadecagon" (2002, 2004). Now, since 2004 there is a formal, mathematically founded theory of all these kinds of structures, a new flourishing branch of logic and geometry, "*N*-Opposition Theory" (for short: "NOT"), also called "oppositional geometry". This general theory of the oppositions among *n* terms shows that after the logical square (n=2) and hexagon (n=3), there is a logical cube (n=4), and that these three oppositional solids belong to an infinite series of "oppositional bi-simplexes of dimension *m*" (in fact, the theory tells much more). Using NOT, in this paper we examine McNamara's "deontic dodecagon", which aims at expressing this author's system DWE (for "Doing Well Enough"), one of the standard models for dealing logically with "supererogation". After showing that, despite the fact that its underlying DWE system is logically sound and complete (as proven by Mares and McNamara in 1997), the oppositional geometry presented as being a "deontic dodecagon" is mistaken (for in NOT's terms this polygon is irremediably both oppositionally redundant and oppositionally incomplete) we show how to correct it, strongly but successfully, within the NOT framework.

Keywords: Deontic logic; Oppositional Geometry; McNamara; DWE; supererogation.

Resumo: Geometria oposicional, isto é, o estudo das "imagens opostas", tem sido estudada de uma maneira bastante randômica. Na lógica deôntica, para além do "quadrado deôntico" clássico (uma aplicação particular do quadrado lógico, ou quadrado da oposição), estudos deram origem em 1972 ao "hexágono deôntico" de Kalinowski (uma aplicação particular do "exágono lógico" de Jacoby, Sesmat e Blanché), ao "decágono deôntico" de Joerden (1987), ao "dodecágono deôntico" e "octodecágono deôntico" de McNamara (1996) e ao "decágono deôntico" e "hexadecágono deôntico" de Wessel (2002, 2004). Atualmente, desde 2004 há uma teoria formal, matematicamente fundamentada, de todos estes tipos de estruturas, um novo e florescente ramo de investigação na geometria e na lógica, "*N*-Opposition Theory" (abreviadamente: "NOT"), também chamada "geometria oposicional". A teoria geral das oposições entre *n* termos mostra que para além do quadrado lógico (n=2) e do hexágono (n=3), há um cubo lógico (n=4), e que estes três sólidos oposicionais pertencem uma série infinita de "bi-simplexos oposicionais de dimensão *m*" (de fato, a teoria diz mais que isso). Ao usar NOT, neste artigo examinamos o "dodecágono deôntico" de McNamara, que expressa o sistema DWE (abreviatura para "Doing Well Enough") do autor e é um dos modelos padrão para o tratamento lógico da "superrogação". Após mostrar que, a despeito de o sistema DWE ser logicamente consistente e completo (conforme demonstrado por Mares e McNamara em 1997), a geometria oposicional apresentada como "dodecágono deôntico" está errada (pois, em termos de NOT, este polígono é irremediavelmente ambos, oposicionalmente redundante e oposicionalmente incompleto), nós mostramos como corrigi-la, de maneira forte porém bem-sucedida, dentro da estrutura NOT.

Palavras-chave: Lógica Deôntica; Geometria Oposicional; McNamara; DWE; superrogação.

* PhD, University of Neuchâtel, Switzerland. Contato: alemore@club-internet.fr

1. The science of the logical-geometrical oppositions: "*n*-opposition theory", alias "NOT" (2004)

What is "*n*-opposition theory"? It is a new mathematical theory comprising Aristotle's "logical square" (or "square of opposition") and Jacoby's, Sesmat's and Blanché's "logical hexagon" as particular cases of a more general logical-geometrical notion, that of "oppositional bi-simplex of dimension *m*". Remind that in Aristotle's "opposition theory" (generating the logical square) there are 4 relations forming the conceptual framework necessary in order to think opposition: contrariety (represented conventionally in blue), subcontrariety (in green), contradiction (in red) and subalternation (in grey)[1]. Jacoby's (1950), Sesmat's (1951) and Blanché's (1953) logical hexagon, despite its innovative power, still respects the quaternality of the opposition relations "package": what changes in it is just the number of opposed terms, 3 instead of 2, but not the number of "kinds" (or colours) of different oppositions (cf. figure 1)[2].

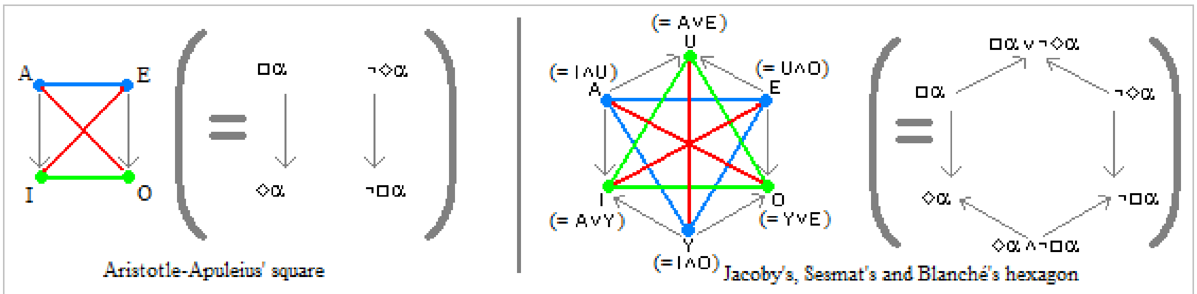

Figure 1. Aristotle-Apuleius' square and Jacoby-Sesmat-Blanché's hexagon: 2-opposition and 3-opposition

Moreover, because the square and the hexagon are constructed upon, respectively, a segment (of contrariety) and a triangle (of contrariety), and because the segment and the triangle are elements of a mathematically well-known series, that of the "geometric simplexes", NOT shows that this generalisation leading from the square to the hexagon can in turn be generalised using the mathematically classical notion of "geometrical simplex of dimension *m*" (cf. fig. 2).

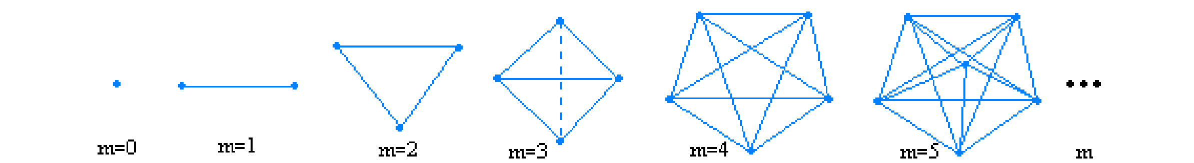

Figure 2. The geometric simplexes of dimension *m* (in blue they express contrarieties)

[1] For a couple of things : "contradiction" (i.e. classical negation) is defined as the impossibility of being both false and the impossibility of being both true; "contrariety" as the possibility of being both false but with the impossibility of being both true; "subcontrariety" as the impossibility of being both false, with the possibility of being both true; "subalternation" (i.e. implication) as the possibility of being both false and the possibility of being both true, with the impossibility of having the first true and the second false (subalternation is an ordered relation, i.e. an arrow going from a first to a second – not so the three other relations, which are symmetric relations).

[2] For having a different number of kinds (or colours) of oppositions, cf. the notion of "oppositional poly-simplex of dimension *m*" (and that of "Aristotelian p^q-semantics"), cf. Moretti [2009 PhD] and Angot-Pellissier [2014?].

As it happens, a "logical bi-simplex (of dimension *m*)" is made of two twin geometrical simplexes (of dimension *m*), one conventionally depicted in blue, for contrary terms: its vertices are, for any couple of them, mutually contrary; and one conventionally depicted in green (the oppositional dual of the first) for subcontrary terms: its vertices are, for any couple of them, mutually subcontrary (cf. fig. 3).

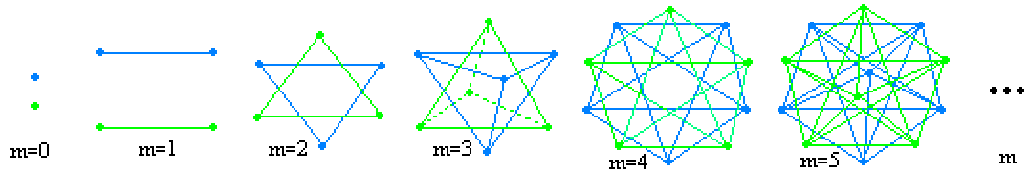

Figure 3. The oppositional bi-simplexes of dimension *m* (the green simplexes express subcontrarieties)

The first striking result of NOT is that, in each case (i.e. for any integer value of *m*, $m \geq 2$) we can build an elegant solid expressing perfectly logical opposition for *n* terms (where $n=m+1$): think of *n* enemies, or *n* competitors. That is, upon contrariety and subcontrariety admitted by construction (as being two logically interpreted geometrical simplexes, blue and green), contradiction is expressed by the lines (conventionally red) connecting symmetric terms by central symmetry (in other words, by red diagonals, cf. fig. 4).

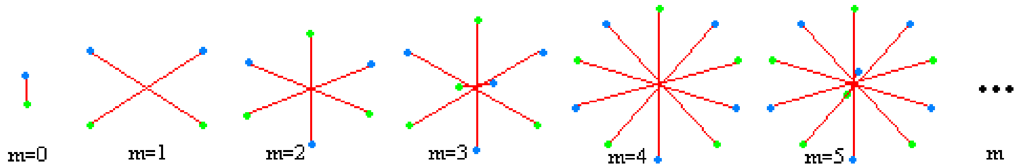

Figure 4. The red diagonals (of central symmetry) express contradictions between (blue) contrary and (green) subcontrary terms

Finally, subalternation (i.e. logical implication) can be expressed by arrows (conventionally grey) starting from *any* blue vertex to *any* green vertex (excepting the one centrally symmetrical to it). The result of this is, in each case (i.e. for any *n*), an elegant oppositional (hyper-)solid, furnished with all the logical properties that Aristotle (or his heirs) had discovered for the square, and that Jacoby, Sesmat and Blanché had proven to hold for the hexagon as well, that is (broadly), the fact that:

1) contradiction is always expressible as central symmetry;
2) there is a duality of the blue and green symmetric oppositional simplexes;
3) there are arrows going systematically from *each* blue point to *each* green one, except for the couples of blue and green points which are mutually centrally symmetric;
4) between any two points, there is one and only one opposition relation.

All these oppositional structures (i.e. the oppositional bi-simplexes of dimension *m*) compose a series of so-called "α*n*-structures" (cf. fig. 5).

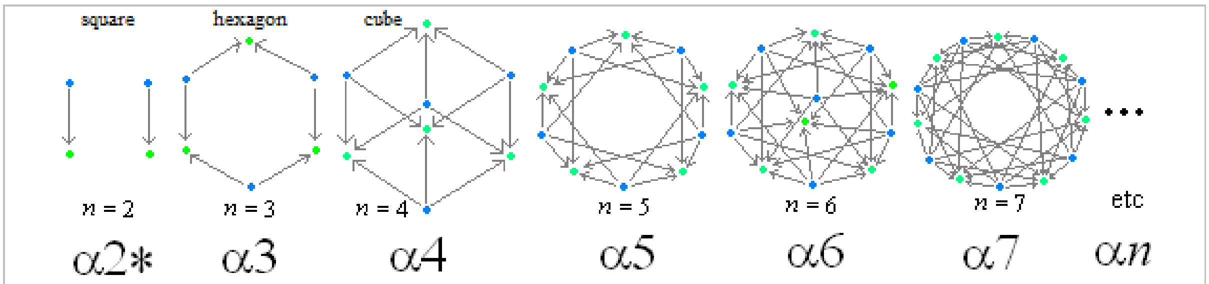

Figure 5. The series of the oppositional bi-simplexes, or *n*-oppositions, or α*n*-structures (*n*=*m*+1)

NOT thus both shows *what* the possible oppositional figures are – for instance, the next figure, after Aristotle's logical square (*i.e.* the oppositional bi-segment) and Jacoby-Sesmat-Blanché's logical hexagon (*i.e.* the oppositional bi-triangle) is the previously unknown "logical cube" (or oppositional bi-tetrahedron) – and *how* to obtain them (again, by a suited interlacing of two dual oppositional simplexes, and by drawing the diagonals and the arrows, cf. Moretti [2004]).

Having shown the mathematical possibility (and thus the existence) of an infinite series of oppositional structures (logical square, logical hexagon, logical cube, …, α*n*-structure, …), conceived as oppositional bi-simplexes of dimension *m*), the next task of the theory is to show that these oppositional structures, infinite in number and in growing logical-geometrical complexity, are useful, that is that they admit meaningful *decorations* (i.e. attributions of logical or semantic or whatever "values"): this oppositional formalism can be used in order to display over solid hyper-space the knowledge about the possible oppositional relations between modalities (*e.g.* the fact that "necessary α" implies – or does not imply – "possible α", and so on). This is done by the introduction of two suited decorating techniques, that of the modal graphs, or γ-structures (cf. Moretti [2004]), and that very powerful (in fact exhaustive), of the modal graphs' "settification" (cf. Pellissier [2008])[3].

This leads us to one of the methodologically most important teachings of the theory, the distinction made (and to be kept!) between "modal graphs" (or γ-structures) and "oppositional structures" (or α- and β-structures). The **modal graph** of any modal system is the diagram showing by points (modalities) and arrows (implications) which dependencies there are between the "basic modalities" of this system (a basic modality is one which cannot be reduced *logically* to the iteration or composition of smaller ones)[4]; whereas the **oppositional structures** (*i.e.* the α*n*-structures and the β*n*-structures) of a modal system are its modal oppositions, *i.e.* the oppositional bi-simplexes of dimension *m* (plus their *closures*, cf. *infra*) which can be decorated by the modalities (basic or not) generated (by compositions using the propositional binary connectives) by the given modal system's modal graph. Note that whereas modal graphs can have almost any possible shape, the oppositional

[3] Using another strategy Hans Smessaert arrives to comparable results (cf. Smessaert [2009]).
[4] For this notion of "basic modality" of a given modal system, cf. B. Chellas, *Modal Logic. An Introduction*, Cambridge University Press, Cambridge, 1980, p. 149, as well as Hughes G.E. and Cresswell M.J., *A New Introduction to Modal Logic*, Routledge, London, 1996, p.55, 56 and 60 . Here we take in consideration only the modal systems of which the number of basic modalities is finite (as is, among others, the case with the modal systems S4, S5 and KD45).

structures belong to two fixed lists, that of the series of the oppositional bi-simplexes of dimension *m* (the α*n*-structures, cf. *supra*) and that of their closures (the β*n*-structures). This distinction between modal graphs and oppositional structures is tricky (slippery), because some mathematical objects, like the logical square, may seem to belong to both. And, as a matter of fact, the two are generally always confused – logicians and philosophers look for geometrical oppositions in a random way – making it impossible to reach, starting from a given modal system, the real *complete* catalogue of the possible oppositions of the logical space of this modal system. As an example, in the case of the Lewis system S5 (the “universal system”, *i.e.* the modal counterpart of classical logic) it can be shown that its modal graph is constituted by six basic modalities (cf. left side of fig. 6), whereas the complete set of the oppositional structures decorated by it (*via* Pellissier’s method, cf. Pellissier [2008]) is represented geometrically by a very regular 3D solid with 14 vertices, 36 edges (arrows) and 24 sides (triangles delimited by arrows), Sauriol’s “logical tetrahexahedron” (cf. right side of fig. 6)[5].

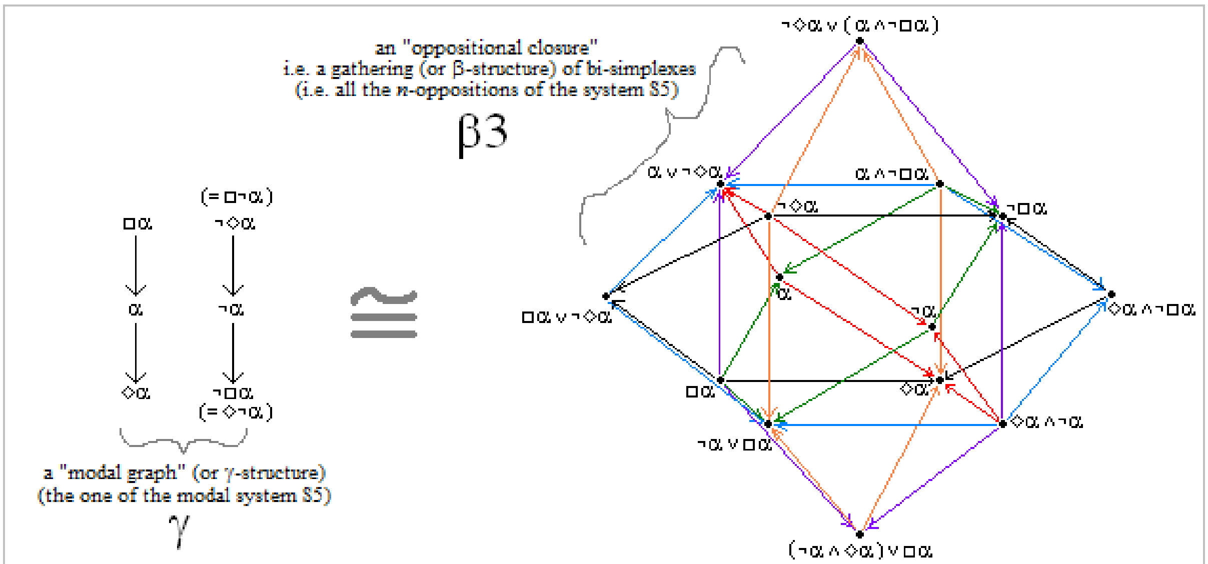

Figure 6. Sauriol's "logical tetrahexahedron" (1968), rediscovered independently by Pellissier (2008) and Smessaert (2009)

Remark on the figure of the logical (or oppositional) tetrahexahedron (i.e. the β3-structure) that the system S5 contains thus, as its oppositional structures (i.e. as its α*n*-structures), not only the well-known logical square (Aristotle’s square) and logical hexagon (Jacoby-Sesmat-Blanché’s hexagon, here, in fig. 6, the black hexagon), but exactly one logical cube (the figure’s core), six logical “strong” hexagons (here in black, blue, red, green, orange, violet) – plus four logical “weak” ones, geometrically non-planar, inside the cube[6] – and 18 logical squares (three for each strong hexagon)!

[5] Using another strategy, Hans Smessaert found independently a similar result in terms of a “rhombic dodecahedron”, which is an equivalent of the logical tetrahexahedron containing all its 14 vertices (it does not highlight the 12 arrows of the logical cube constituting the tetrahexahedron’s heart – but they can be deduced by transitivity of the rhombic dodecahedron’s arrows), cf. Smessaert [2009].

[6] For this distinction between “strong” and “weak” geometrical figures (i.e. oppositional solids), cf. Pellissier [2008] and Angot-Pellissier [2012] (for short, in weak figures some logical equivalencies become logical implications; geometrically speaking, such figures are “broken”, i.e. they do not belong anymore to a single plane, but to three ones two by two perpendicular). Here we only deal with the strong figures (with the exception

So, not one logical square (Aristotle), neither three (as classically inside Jacoby's, Sesmat's and Blanché's logical hexagon), but 18 ones…

Among many other useful results of NOT, the following three must be recalled here (at least briefly) before we turn to an examination of the deontic polygon of McNamara (the dodecagon). Firstly[7], Pellissier's set-theoretical method allows for any finite *linear* modal graph, no matter how big) to reduce it to a characteristic set "E". This set, by a suited set-theoretical "partition technique" displayed over it, allows making a complete list of the possible oppositional structures of the starting modal system. Secondly, the oppositional structures (i.e. the oppositional bi-simplexes: logical squares, hexagons, cubes, …) thus found in each case support a higher geometrical ordering, namely that of the series of the oppositional closures or β*n*-structures (cf. Moretti [2009 PhD] and Pellissier [2008]) (cf. fig. 7).

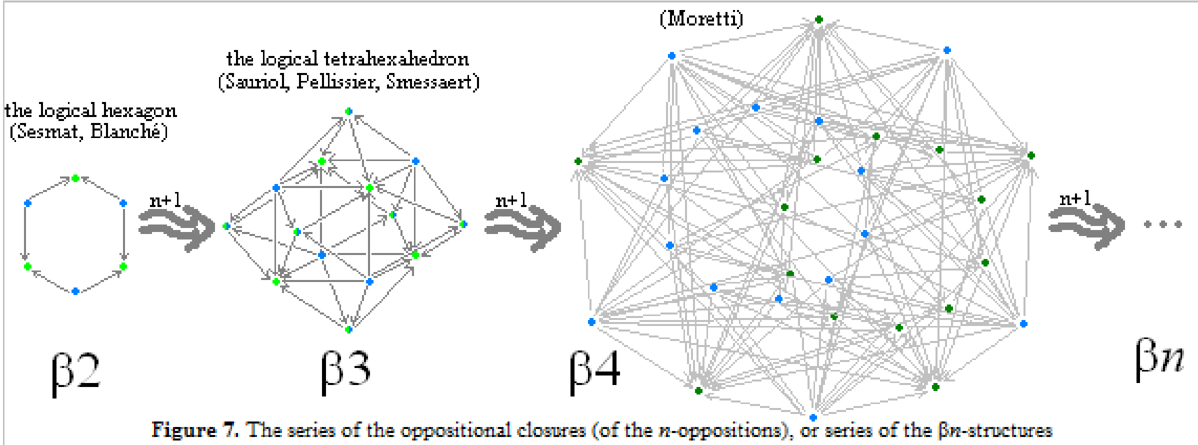

Figure 7. The series of the oppositional closures (of the *n*-oppositions), or series of the β*n*-structures

This series can be shown to be weakly fractal, for the figure of each term of the series contains as parts the figures of the previous terms[8]. The three-dimensional oppositional tetrahexahedron depicted above for the Lewisian system S5 (cf. fig. 6 *supra*) is only one of its terms (as we said, it is the β3-structure).

Thirdly, a powerful result shows that modal graphs different in shape can happen to generate the same characteristic set "E" (cf. Pellissier [2008]). Which means that such "different" graphs can be turned into different decorations of the same abstract β*n*-structure (or oppositional closure), which in turn means that they are in some (new) respect oppositionally equivalent (this generates a new kind of logical-geometrical equivalence class). Thus NOT seems to offer to logic a new space of possible translations.

of the logical square, which itself is – as Pellissier has demonstrated, a weak 2-opposition, cf. Angot-Pellissier [2012]), which are the standard ones.

[7] Cf. Pellissier [2008].

[8] The fractal behaviour is made particularly clear in the study of a remarkable fragment of the series of the β*n*-structures, namely the so-called series of the "oppositional hyper-flowers", cf. Moretti [2009 PhD], § 11.04.02-03.

Having briefly recalled these elements (for more ones and for more detailed ones cf. Moretti, "Why the Logical Hexagon?"), we can get closer to the proper subject of the present paper.

2. Known results on the exhaustive oppositional geometry of standard deontic logic

Here we recall, in an outrageously brief way, the results we established in Moretti [2009] about the exhaustive oppositional geometry of standard deontic logic. The main point to be grasped is that the oppositional geometry of standard deontic logic (i.e. the counterpart of the modal system KD45) has been perfectly identified inside NOT: it is an oppositional-geometrical space bigger than that of alethic modal logic (i.e. S5), but smaller than those, respectively, of epistemic (i.e. S4) and tense logic (cf. Moretti [2009 PhD], ch. 17). In NOT's terms it is equivalent to the β5-structure (the 5-dimensional oppositional hyper-tetrahexahedron): it is a 5-dimensional solid, gathering α*n*-structures (oppositional bi-simplexes decorated deontically), with $n \in \mathrm{N}$, $2 \leq n \leq 6$.

2.1. The non-linear modal graph of standard deontic logic

The identification of this oppositional-geometrical space has been made following the methodology of NOT. Firstly, an examination has shown that the "basic modalities" of standard deontic logic (i.e. KD45) are six: O*p*, *p*, ∇*p*, ¬O*p*, ¬*p*, ¬∇*p*[9]. Secondly, the examination of their mutual relations, i.e. their spatial diagrammatical disposition inside the (non-linear) "modal graph" of standard deontic logic, has shown that, differently from alethic modal logic, there are here "isolated points" (the null modalities *p* and ¬*p*), which are said to be "isolated" insofar no arrow of the modal graph (or γ-structure) touches them (cf. fig. 8)[10].

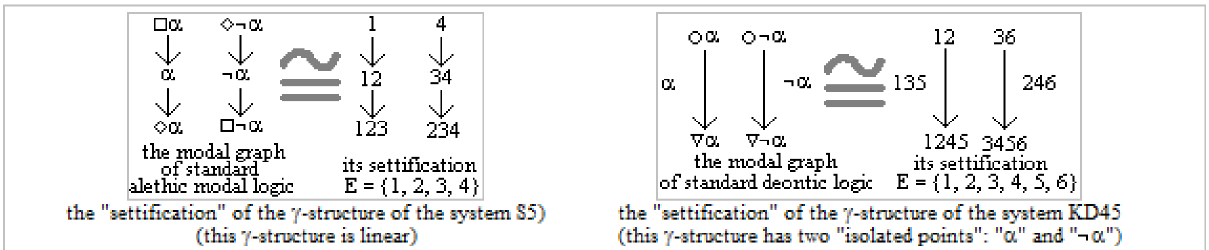

Figure 8. Comparing the application of "settification" to the modal graph (or γ-structure) of the system S5 to that of the system KD45

Thirdly, the numbering method, duly adapted by us to the non-linear case of the isolated points, has shown with respect to the system KD45 that the characteristic Pellissier set "E" of this deontic logic is E = {1, 2, 3, 4, 5, 6} (cf. fig. 8 *supra*, right side).

9 "O*p*" ≡ "*p* is obligatory", "*p*" ≡ "it is the case that *p*", "∇*p*" ≡ "*p* is permitted".
10 This expresses the specific *deontic* flavour, i.e. the fact that "obligatory" does not imply "real" and that "real" does not imply "permitted" (cf. right side of fig. 8), whereas in alethic modal logic "necessary" implies "real" and "real" implies "possible" (cf. left side of fig. 8).

2.2. The bi-simplicial components of the oppositional space of standard deontic logic

The translation between deontic basic modalities and symbols (here numbers) composing strings is given by the equation table between the deontic modal graph and its numbered version (cf. the right part of fig. 8 *supra*)[11]. Starting from this, and knowing that *n*-opposition equals (*via* Pellissier's translation) set-theoretic *n*-partition – the sets of strict substrings of E represent possible α*n*-structures –, it suffices to look for all the *n*-partitions of the set E: this will give all the oppositional bi-simplexes (of dimension *n*-1) contained in the standard deontic space. The combinatorial result (in Moretti [2009] we detail it and study it with many figures) is the following: in the modal system KD45 there are one 6-partition (a deontic α6-structure), fifteen 5-partitions (deontic α5-structures), sixty-five 4-partitions (deontic cubes), ninety 3-partitions (deontic hexagons) and 270 deontic squares (cf. fig. 9).

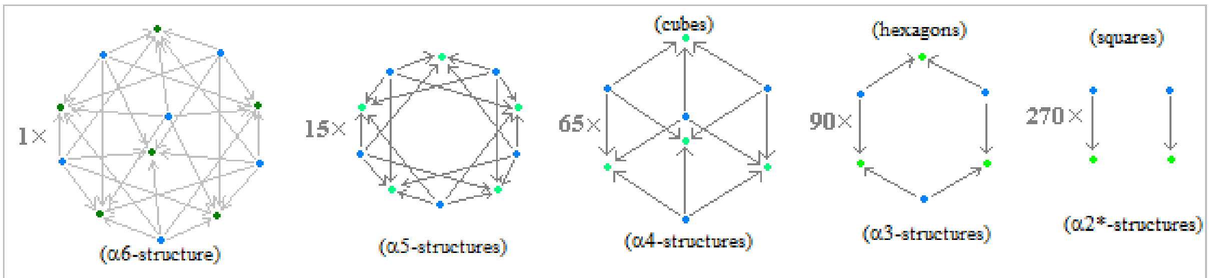

Figure 9. The oppositional α*n*-structures of the standard deontic space (Moretti, 2009)

So, the deontic oppositional geometry is much richer than it was thought, that is: it is much richer than just one "deontic square" or just one "deontic hexagon".

2.3. The 5-dimensional solid of standard deontic logic

Graphically speaking, the oppositional core of the standard deontic space (i.e. the system KD45) can be represented as a β5-structure decorated deontically (in Moretti [2009] we show how exactly). This implies that in KD45 there are different kinds of deontic β*n*-structures: actually, all (and only) the ones contained in the β5-structure (cf. fig. 10).

[11] The table only gives the numbers for the 6 basic modalities (for instance: "Oα" is equivalent to the string "12", "α" is equivalent to the string "135", etc.). All the other possibilities, all the other modalities, obtainable by Boolean combinations of the 6 basic ones (like "α∧Oα", etc.), are reachable from the given translation by Boolean compositions of the corresponding strings (of numbers): "∨" corresponds to concatenation, "∧" to intersection and "¬" to complementation with respect to the set E: for instance, "α∧Oα" is equivalent to the string "135∩12", and therefore to the string "1" (cf. Moretti [2009]).

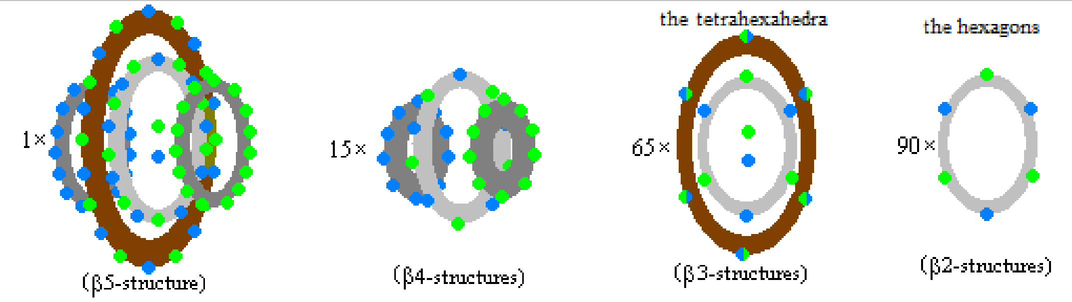

Figure 10. The β*n*-structures (here simplified graphically speaking) making the standard deontic space (Moretti 2009)

Having (very, very briefly!) recalled some of the main theoretical data concerning the exhaustive oppositional geometry of standard deontic logic, as it is accessible *with* the help of NOT, we can now turn to our present question, the examination of the classical geometrical approaches made, *without* knowledge of NOT, about deontic logic.

3. The “deontic polygons” proposed, beyond the deontic square, in the known literature

As long as we know, there have been at least four scholars who have made proposals in order to complexify the geometrical characterisation of deontic logic.

3.1. Kalinowski’s (1972), Joerden’s (1987) and Wessels’ (2002, 2004) deontic polygons

The first oppositional-geometrical reform of the deontic square seems to have been that proposed by the Polish-French logician and philosopher Georges (Jerzy) Kalinowski, who has shown in 1972 (cf. Kalinowski [1972] and [1996]) that Sesmat’s and Blanché’s logical hexagon (cf. *supra*) can be applied perfectly to deontic logic (which Kalinowski developed independently from G.H. von Wright). This gives a “hexagon of norms” (cf. fig. 11). This conservative extension of the deontic square is fine but, as we now know (cf. *supra*), it is not geometrically exhaustive of standard deontic logic, for the geometry of the latter is not 2-dimensional, but 5-dimensional. In particular, as we just saw, Kalinowski’s deontic hexagon does not take into account the very important deontic “null modalities”, namely “*p*” and “¬*p*”, that is: “*p* is real”, “*p* is not real” (cf. fig. 11).

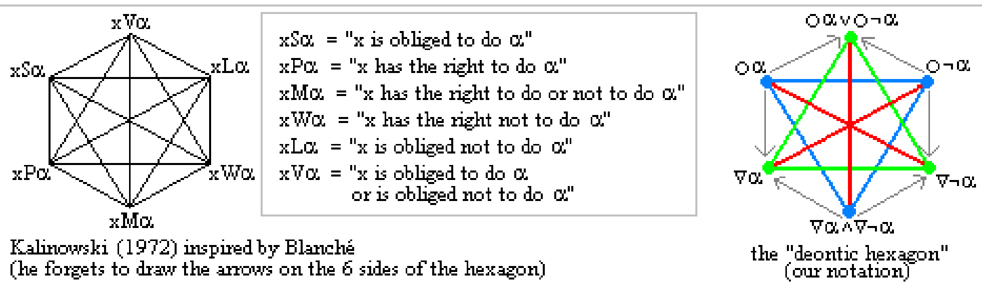

Figure 11. Kalinowski's "hexagon of norms" (1972)

In this respect Kalinowski's deontic hexagon is only a fine fragment of the whole standard deontic geometry (in Pellissierian terms it is the tripartition "12|36|45" of E).

A second interesting proposal has been made in 1987 by the German lawman and logician Jan C. Joerden (in a joint paper with J. Hruschka, cf. Hruschka and Joerden [1987]; cf. also Joerden [2012]), while trying to express logically the subtle, philosophically very important concept of "supererogation": "doing more (good) than it is demanded (as a minimum)" (cf. fig. 12).

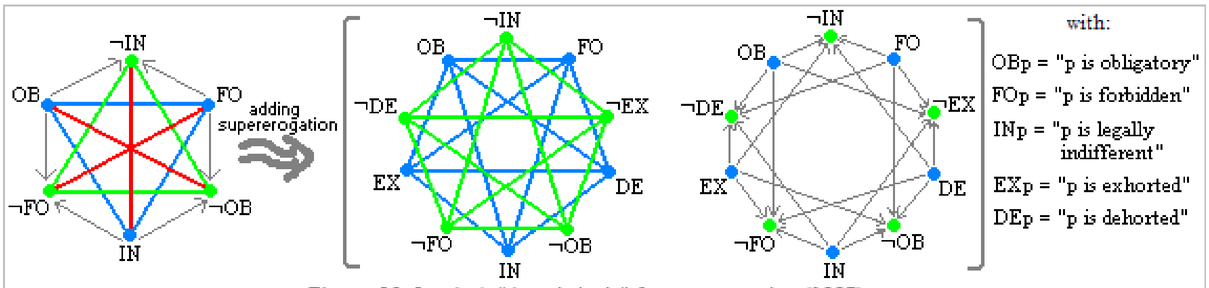

Figure 12. Joerden's "deontic logic" for supererogation (1987)

Another interesting proposal, also aiming at formalising logically "supererogation", has been done more recently by the German philosopher and logician Ulla Wessels (cf. Wessels [2002] and [2004]) (cf. fig. 13).

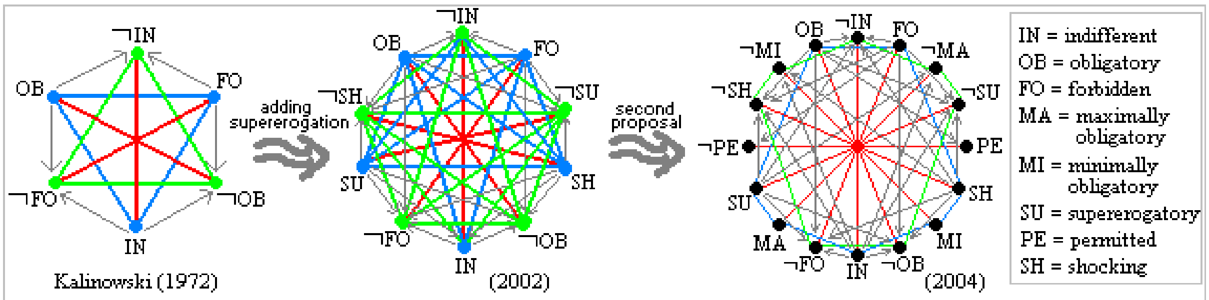

Figure 13. Wessels' "deontic decagon" (2002) and "hexadecagon" (2004) for supererogation

We will not analyse here these proposals (it would be interesting to do it elsewhere). Let us turn instead toward a fourth, very interesting geometrisation of deontic logic, one which in some sense has become standard for expressing supererogation.

3.2. McNamara's attempts to think geometrically deontic logic beyond the deontic square: his deontic hexagon, dodecagon and octodecagon (1996)

Another author has tried to go beyond the classic deontic square. By a reasoning seemingly independent from Jacoby's, Sesmat's and Blanché's logical hexagon (and *a fortiori* from Kalinowski's hexagon of norms), the American philosopher and logician Paul McNamara has proposed a "deontic hexagon" (cf. McNamara [2006]). As we show in another place (cf. Moretti [2014?]), this structure, which is not a logical bi-simplex, and which seemingly is rather different from Kalinowski's "hexagon of norms", is both redundant (for its vertices "OP*p*" and "OP¬*p*" are in fact the same, as recognises McNamara himself) and incomplete, for "OP*p*" has not its contradictory term, "¬OP*p*", on the polygon (cf. fig. 14 and 15).

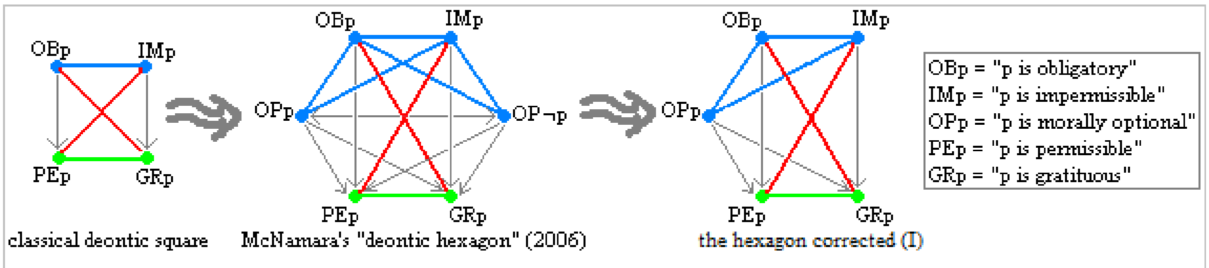

Figure 14. Correcting McNamara's "deontic hexagon" from redundancies

Now, after eliminating from it the oversight of ¬OP*p* (cf. fig. 15, the two structures in the middle), we get a logical bi-simplex (of dimension 2) which is nothing less (and nothing more) than the real "deontic hexagon", strictly equivalent to Kalinowski's one (cf. fig. 15, fourth structure on the right)[12].

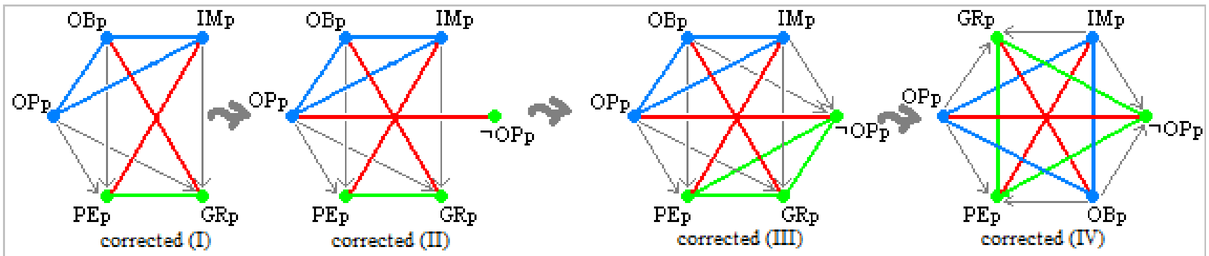

Figure 15. Correcting McNamara's "deontic hexagon" from oversights

But the point we want to concentrate on in our paper is rather the following. In 1996, while developing a very interesting "logic of common-sense morality" (cf. McNamara [1996a] and McNamara [1996b]), an enriched version of deontic logic (obtained, essentially, by building a coherent framework for the modality "supererogation"), McNamara proposed two new graphs (two new "deontic polygons"). In the rest of this paper we study the first of them, the so-called "deontic dodecagon" (cf. fig 16)[13].

[12] One only has to rotate it properly (i.e. anticlockwise) of 90° and then to take the vertical mirror image (i.e. exchanging left with right) in order to get back Kalinowski's deontic hexagon (graphical exercise left to the reader!).

[13] The second one, the "deontic octodecagon", is shown in figure 27 *infra*.

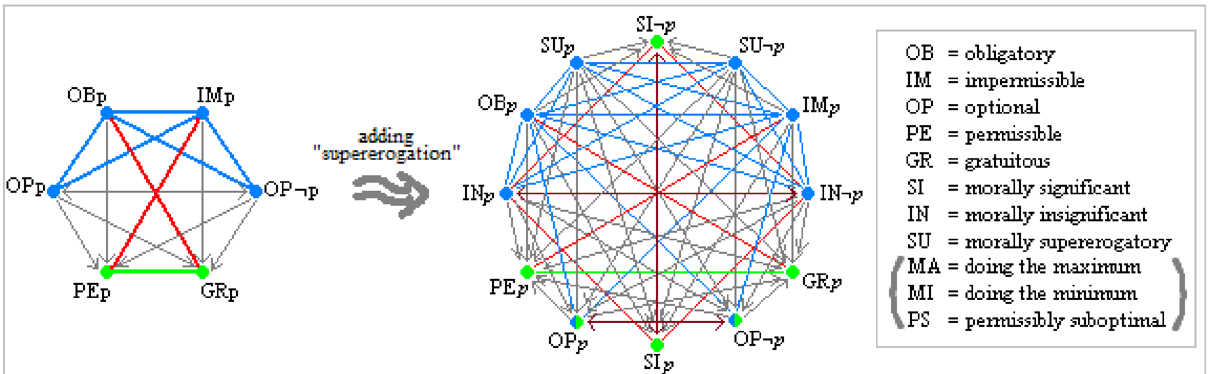

Figure 16. McNamara's "deontic dodecagon" for supererogation (1996)

In what follows, and similarly with what we saw for his mistaken "deontic hexagon", we will show that, despite the intrinsic great philosophical interest (and full logical correction, cf. Mares and McNamara [1997]) of McNamara's deontic-ethical system DWE (meaning "Doing Well Enough"), one of its two geometrical expressions (the deontic dodecagon), which is neither a logical bi-simplex (an αn-structure) nor a gathering of logical bi-simplexes (a βn-structure, i.e. an oppositional closure), suffers both of redundancies and of oversights, so that one can and must correct it accordingly. This gives, once all the redundant terms are eliminated, once the omitted terms are added and once the vertices are duly rearranged in space (guided by the rule of the central symmetry of the contradictories), a quite complex, but elegant and interesting solid (not a polygon) much more regular than the starting one, coherent with oppositional geometry (if not yet exhaustive), which shows that McNamara's philosophical interesting analysis is also geometrically viable (in the sense of oppositional geometry).

4. Correcting McNamara's "deontic dodecagon"

In this section we want to go beyond McNamara's geometrisation of his own deontic-modal ideas, in order to show that the latter deserve a better geometrical-oppositional representation. In order to do this the first task consists in revealing the possible mistakes contained in his deontic dodecagon. Only thereafter it will be possible to get near to a better solution, fully compatible with oppositional geometry.

4.1. Redundancies and oversights with respect to NOT

If, as we have seen, the examination of McNamara's hexagon suggests to be careful (for he had to work without the help of oppositional geometry, which didn't exist by that time), a direct examination of the deontic dodecagon confirms such methodological doubts: there are at least four reasons, from the point of view of oppositional geometry, to be dissatisfied with the dodecagon.

First, some terms are present two times in the structure. These are: IN¬*p* (for it is logically equivalent to IN*p*, already present in the deontic dodecagon), SI¬*p* (for it is logically equivalent to SI*p*, already present in the dodecagon), OP¬*p* (for it is logically equivalent to OP*p*, already present)[14]. These are undue geometrical redundancies.

Second, eight relations, among all the possible oppositional ones between the 12 vertices of the dodecagon, are undefined: the PE*p*----SI*p*, SI*p*----GR*p*, GR*p*----SI¬*p*, SI¬*p*----PE*p* and the OP*p*----SI*p*, SI*p*----OP¬*p*, OP¬*p*----SI¬*p*, SI¬*p*----OP*p* (cf. fig. 17).

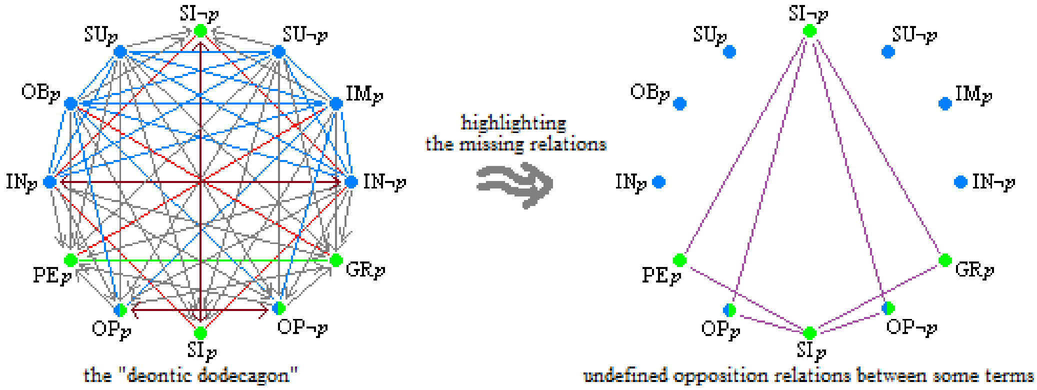

Figure 17. The "deontic dodecagon" and its 8 undefined opposition relations

Now, this omission is neither necessary nor tolerable: this dodecagon is supposed to be an oppositional-geometric figure, and in *each* oppositional-geometric figure (so teaches us oppositional geometry, following Aristotle and Sesmat-Blanché) each point is related to any other point by some kind of opposition relation (among Aristotle's four possible ones). So, if the deontic dodecagon is to represent something (i.e. if McNamara's analyses are to be taken seriously – as they clearly are), these missing relations must in fact exist somewhere (it must be possible to complete this drawing within this respect).

Thirdly, a second kind of omission, not of relations but of terms (i.e. vertices) this time, is present in the structure. For three terms (three contradictory negations) are clearly missing: the ¬OP*p*, ¬SU*p* and ¬SU¬*p*, exactly (contradictory of the OP*p*, SU*p* and SU¬*p* respectively).

Fourthly and finally, some couples of contradictory terms (vertices), although both (the term and its contradiction) present in the figure, are not *two by two centrally symmetric* (the one with respect to the other): this is the case of the IN*p* and SI*p* (the red contradiction relation uniting them is not one of the diagonals of the dodecagon, as it should be according to oppositional geometry).

The deontic dodecagon is thus clearly inadequate in order to be an oppositional geometrical-logical structure, it has to be corrected, McNamara's philosophical and logical very interesting ideas deserve another, better geometrical representation.

[14] This results from McNamara's own axioms: ION, IIN and by the definition of SI.

4.2. Eliminating the redundancies in the deontic dodecagon

The starting deontic dodecagon is a complex structure, with very few regularities if we except a left-right symmetry. In fact, if we eliminate the three redundant terms (i.e. SI¬*p*, OP¬*p* and IN¬*p*), the structure becomes clearer, if not yet richer of symmetries (cf. fig. 18).

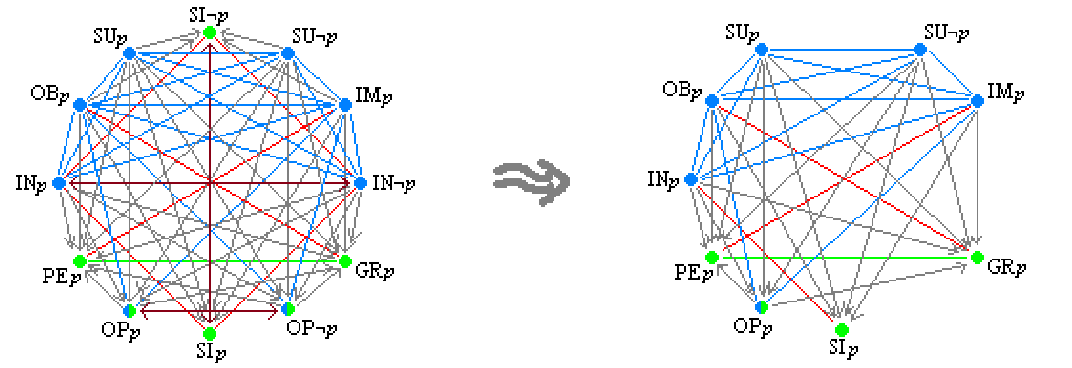

Figure 18. Eliminating the redundancies of McNamara's "deontic dodecagon"

As we did it previously with McNamara's deontic hexagon, we need here to go further, in the hope of reaching finally an oppositional polygon or (hyper-)solid faithful to McNamara's ideas and acceptable from the point of view of oppositional geometry.

4.3. Eliminating the oversights in the deontic dodecagon

So, we need now to restore the forgotten elements, that is three missing contradictory negations: ¬SU*p*, ¬SU¬*p* and ¬OP*p*; and we need to determine the kind of opposition embodied by each of the 8 aforementioned missing relations (fig. 17 *supra*). But before adding them it is useful to modify slightly (by harmless permutations of the vertices) the so far reduced dodecagon (cf. fig. 19, left side)[15]. If we make momentarily abstraction of the arrows, this gives finally the following rearrangement where two logical bi-simplexes, absent in McNamara's analysis, do appear (cf. fig. 19, right side).

[15] The contradictory negation of SU¬*p* should be, geometrically speaking (i.e. for reasons of central symmetry of the contradictories) in the place actually occupied by OP*p*: where to place this one, then? Because in any case we must restore the central symmetry of IN*p* and SI*p* (which are mutually contradictory) we will move SI*p* at the top place left empty by the elimination of the redundant SI¬*p*: accordingly, we will place its contradictory, IN*p*, at the bottom place. So that now we can move OP*p* at the left side place: then the natural place for its contradictory will be the right side place, left empty by the elimination of the redundant term IN¬*p*. All this leaves us free, now, to put the last missing terms, ¬SU*p* and ¬SU¬*p*, exactly in the places centrally symmetric to those of their respective contradictory terms, namely SU*p* and SU¬*p* (cf. fig. 19, left side).

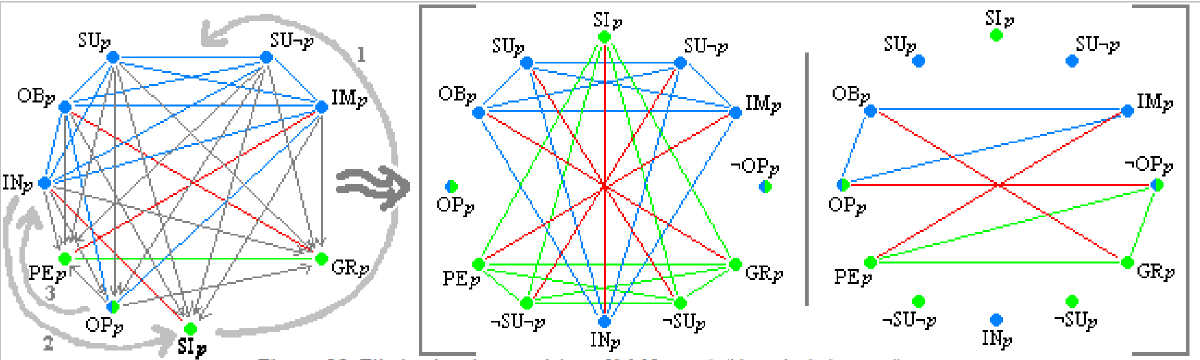

Figure 19. Eliminating the oversights of McNamara's "deontic dodecagon"

4.4. Interpreting rightly the figure thus obtained

This figure, once the arrows restituted (with respect to the central and right parts of fig. 19), is better than the original one, for it respects the geometrical laws of opposition: (1) there is a central symmetry with respect to contradiction; (2) all contradictory terms are drawn (there are no oversights); (3) there are no redundancies. As we can see, it is constituted of two α*n*-structures: an oppositional bi-simplex of dimension 4 and an oppositional bi-simplex of dimension 2 (cf. fig. 20).

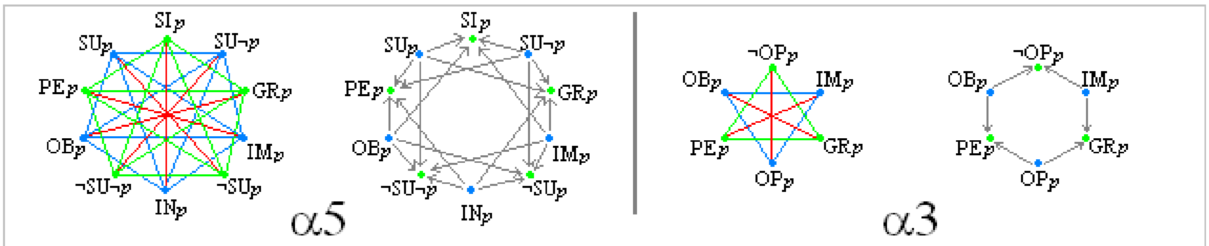

Figure 20. The first two "crypto-McNamarian" deontic bi-simplexes (of dimension 4 and 2 respectively)

If we superpose these two components (under their previous representation of fig. 19, but representing this time also all the known arrows) we get some kind of "new McNamarian NOT-dodecagon" (cf. right side of fig. 21).

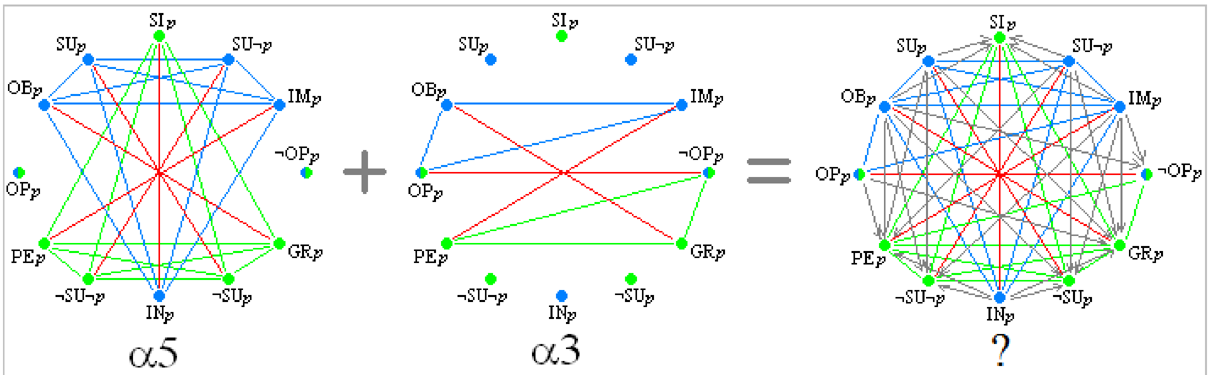

Figure 21. The oppositionally-geometrically corrected McNamarian deontic dodecagon

This figure turns out to be useful, for it shows that so far this (new) polygon is still not complete from the point of view of NOT. As it happens, a close examination shows that some of its relations are still undetermined (cf. the right side of fig 22, in violet).

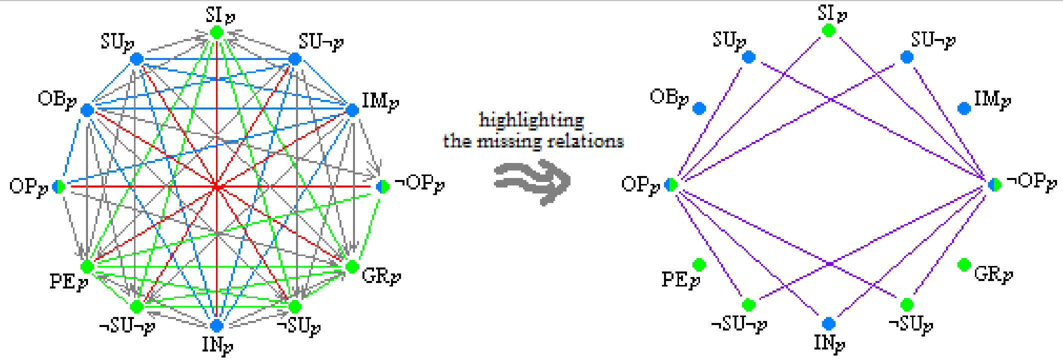

Figure 22. The relations missing in the corrected McNamarian dodecagon

In particular, among the 12 violet now missing relations we see that one (the "OP*p*---SI*p*") had been missing since the beginning (this is the orange one in the next figure 23, undefined in the original deontic dodecagon of figure 17). The remaining 8 violet relations of figure 23 are new: for they concern logical modalities (like "¬OP*p*" or "¬SU¬*p*") which were absent in the starting dodecagon. On the contrary, some of the now missing opposition relations (of figure 22) were already present (i.e. specified) before (cf. fig. 16 *supra*): this is notably the case with the three implications SU*p*→OP*p*, SU¬*p*→OP*p* and IN*p*→OP*p* (depicted in grey at the right side of fig. 23).

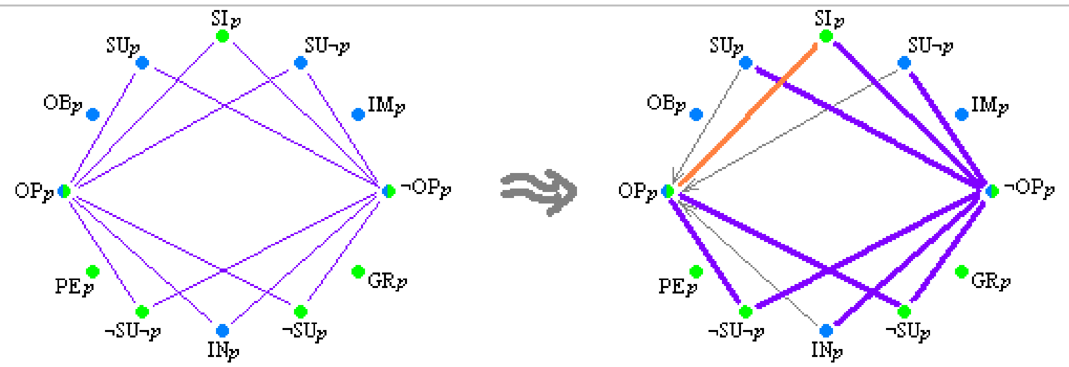

Figure 23. Specifying the quality of some undefined relations

But in fact, taking these three implication arrows (cf. fig. 23) into account, one can easily see (simply by contraposition, i.e. by considering the three new implications ¬OP*p*→¬SU*p*, ¬OP*p*→¬SU¬*p* and ¬OP*p*→¬IN*p* respectively) that each of them generates a new logical (deontic) square (cf. fig. 24).

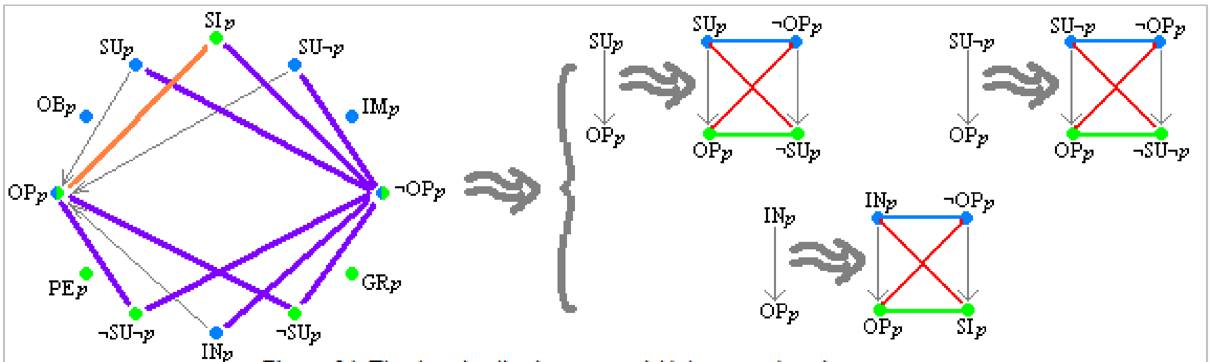

Figure 24. The three implication arrows yield three new deontic squares

Then, it is easy to see that these three terms contrary to ¬OP*p* (i.e. the IN*p*, SU*p* and SU¬*p*), being also mutually contrary (cf. left side of fig. 21 *supra*), form with ¬OP*p* a blue tetrahedron of contrariety, whereas their 4 contradictory negations (i.e. OP*p*, SI*p*, ¬SU*p* and ¬SU¬*p*) form dually a green tetrahedron of subcontrariety. So, by virtue of NOT's theory of the oppositional bi-simplexes, the whole of these 8 points, shaping a bi-tetrahedron, lets emerge a logical (deontic) cube (cf. fig. 25).

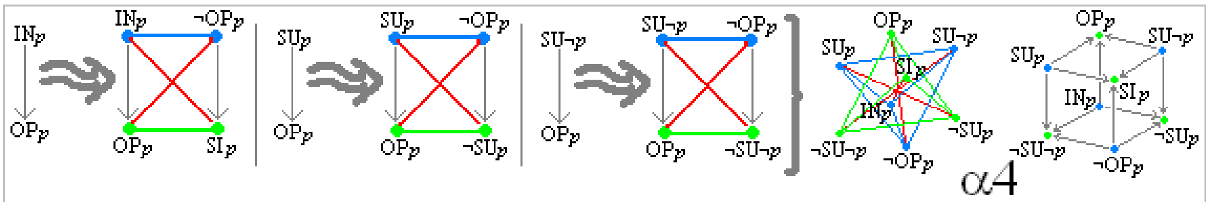

Figure 25. Finding the missing McNamarian third oppositional (deontic) bi-simplex (of dimension 3): a cube

But all this represents a big change: for one can check now that all the 9 previously unidentified violet and/or orange opposition relations (cf. fig. 23, right side) are present and thus well identified through this new "crypto-McNamarian" deontic cube (cf. fig. 25), which means that this time we have, for the 12 vertices we are speaking of, a *complete* oppositional-geometrical knowledge of all their possible opposition relations.

So, what we can and must say finally is that McNamara's "deontic dodecagon" is in fact a complex deontic *hyper-solid* (not a polygon) made of a disguised entanglement of three logical bi-simplexes (of dimension 4, 3 and 2 respectively). The final mixed structure is formed of an α5-structure plus an α4-structure and an α3-structure, the three being partially interlaced (cf. fig. 26).

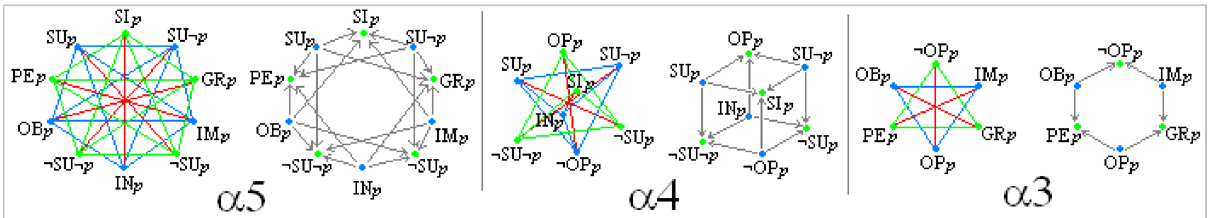

Figure 26. Three McNamarian deontic bi-simplexes (of dimension 4, 3 and 2)

This situation is not new to NOT (cf. Moretti [2009 PhD]), except that inside this conceptual framework such a layout (i.e. a small and incomplete amount of entangled oppositional bi-simplexes) can only be observed relatively to *fragments* of known bigger structures. This suggests, then, that this mixed structure is a fragment of a bigger one, one being perfectly symmetrical, i.e. one respecting the format of oppositional geometry. Put in another way, NOT tells us that here some terms must actually be still invisible (the underlying oppositions cannot be expressed as they are). Remark that this is coherent with our knowledge of the fact that the oppositional geometry of standard deontic logic is summarised by the 5-dimensional β5-structure (cf. Moretti [2009]): so its conservative extensions (as the ones McNamara proposes here) cannot be expressed by spaces geometrical smaller than this. And, as it happens, McNamara's own axiomatisation itself tells already that the logic corresponding to his deontic dodecagon can (and must) be expanded so to have, logically, a more powerful "deontic octodecagon" (cf. fig. 27).

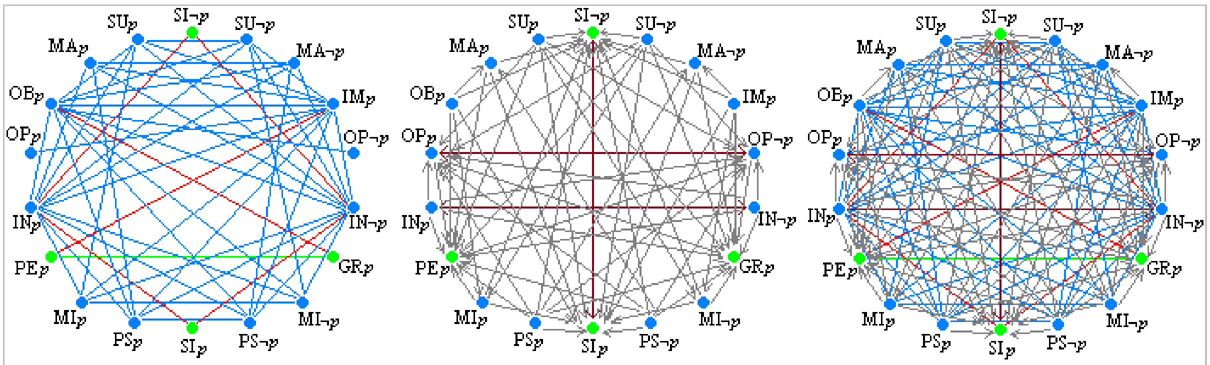

Figure 27. McNamara's "deontic octodecagon" (1996), to be analysed in the future by means of oppositional geometry

For doing this, new modalities representing further deontic-ethical notions are added (refining the conceptual framework of the notion of "supererogation"), in a way such that some of the previous modalities become now derivable from compositions of some of the new ones (conceived as more basic). As an example, McNamara introduces the new modal operators PS*p*, MI*p*, MA*p* (but not their negations, so he introduces, again, oversights in his "polygon") and he defines[16]: SU*p* ≡ PE*p* ∧ MI¬*p*. Examining this, the aim of NOT will be to determine the general "crypto-McNamarian" modal graph of DWE, individuating the general β*n*-structure corresponding to DWE (with, presumably, *n*>5). But the case of this McNamarian "deontic octodecagon" remains to be studied in another paper.

5. Towards the McNamarian deontic β*n*-structure

Summing up, the real geometrical expression of the first half of McNamara's deontic ideas (internalising supererogation) is not a 2-dimensional polygon (a dodecagon), but a hyper-solid made of a 4-dimensional α5-structure (a deontic bi-simplex of dimension 4) intertangled with a 3-dimensional

[16] For the reading of such new modalities, cf. the table in figure 16.

α4-structure (a deontic cube) and a 2-dimensional α3-structure (a deontic hexagon). In order to have the final word of this interesting story we will have to go further and face an examination, similar to the one we developed here, of McNamara's deontic ideas on supererogation, correcting it and showing which oppositional solid, more than 5-dimensional and containing the three logical bi-simplexes we found here, is at stake finally[17].

References

Angot-Pellissier, R. "2-opposition and the topological hexagon", in: Béziau J.-Y.; Payette G. [2012].
_______. "Many-valued logical hexagons in a 3-oppositional trisimplex", (submitted, 2014?)
Bailhache, P. *Essai de logique déontique*, Vrin, Paris, 1991.
Béziau, J.-Y. "New light on the square of oppositions and its nameless corner", *Logical Investigations*, 10, 2003.
_______; Payette G. (eds.). *The Square of Opposition. A General Framework for Cognition*, Peter Lang, Bern, 2012.
Blanché, R. "Sur l'opposition des concepts", *Theoria*, 19, pp. 89-130, 1953.
Chellas, B. *Modal Logic. An Introduction*, Cambridge University Press, Cambridge, 1980.
Gardies, J.-L. *Essai sur la logique des modalités*, PUF, Paris, 1979.
Hruschka J.; Joerden J.C. "Supererogation: Vom deontologischen Sechseck zum deontologischen Zehneck, Zugleich ein Beitrag zur strafrechtlichen Grundlagenforschung", *Archiv für Rechts und Sozialphilosophie*, 73, 1, pp. 93-123, 1987.
Hughes G.E.; Cresswell M.J. *A New Introduction to Modal Logic*, Routledge, London, 1996.
Jacoby, P. "A triangle of opposites for types of propositions in Aristotelian logic", *New Scholasticism*, 24, 1950.
Joerden, J.C. "Deontological Square, Hexagon, and Decagon: A Deontic Framework for Supererogation", *Logica Universalis*, 6, 1-2, 2012.
Kalinowski, G. *La logique des normes*, PUF, Paris, 1972.
_______. *La logique deductive. Essai de presentation aux juristes*, PUF, Paris, 1996.
Luzeaux D. Sallantin J.; Dartnell C., "Logical extensions of Aristotle's square", *Logica Universalis*, 2, 1, 2008.
Mares E.; McNamara P. "Supererogation in Deontic Logic: Metatheory for DWE ans Some Close Neighbours", *Studia Logica*, 57, pp. 397-415, 1997.
McNamara, P. "Making Room for Going Beyond the Call", *Mind*, Vol. 105, 419, July 1996 (a).
_______. "Doing Well Enough: Toward a Logic for Common-Sense Morality", *Studia Logica*, Vol. 57, N. 1, pp. 167-192 (26), July 1996 (b).
_______. "Deontic Logic", *Stanford Encyclopedia of Philosophy*, 2006 (internet).
Moretti, A. "Geometry for Modalities? Yes: Through *N*-Opposition Theory", in: Béziau J.-Y., Costa Leite A.; Facchini A. (eds.). *Aspects of Universal Logic*, University of Neuchâtel, Neuchâtel, 2004
_______. *The Geometry of Logical Opposition*, PhD Thesis, University of Neuchâtel, Switzerland, 2009 (internet).
_______. "The Geometry of Standard Deontic Logic", *Logica Universalis*, 3, 1, 2009.
_______. "Why the Logical Hexagon?", *Logica Universalis*, 6, 1-2, 2012.
_______. "How to Correct a Suboptimal Deontic Hexagon", (submitted, 2014?).
Pellissier, R. ""Setting" *n*-opposition", *Logica Universalis*, 2, 2, 2008
Sauriol, P. "Remarques sur la théorie de l'hexagone logique de Blanché", *Dialogue*, 7, 1968.
Sesmat, A. *Logique II. Les raisonnements, la logistique*, Hermann, Paris, 1951.
Smessaert, H. "On the 3D visualisation of logical relations", *Logica Universalis*, 3, 2, 2009.
_______. "The Classical Aristotelian hexagon versus the Modern Duality hexagon", *Logica Universalis*, 6, 1-2, 2012.
_______; Demey L. "Logical Geometries and Information in the Square of Opposition", (submitted).
Wessels, U. *Die gute Samariterin. Zur Struktur der Supererogation*, Walter de Gruyter, Berlin, 2002.
_______. "Und es gibt doch Supererogationslöcher. Eine Erwiderung auf Jan C. Joerdens Besprechung von *Die gute Smariterin*", *Annual Review of Law and Ethics*, Band 12, 2004.

[17] I wish to thank warmly Paul McNamara for some discussions on several important parts of the general subject of this paper. I owe to Frédéric Sart the knowledge that the deontic modal graph I studied in 2009 is in fact that of the system called "KD45".